→ Flux Expulsion and Homogenization – Non-Ideal, cont'd

– so far, have encountered:

→ S-P reconnection ⇒ weak dissipation ($R_m \gg 1$) has strong effect at singularity – BOUNDARY LAYER

→ Taylor Hypothesis ⇒ small flux tubes destroyed by stochasticity, leaving $\int d^3x\, \underline{A}\cdot\underline{B}$ as robust invariant

∴ diffusive dissipation most effective at breaking freezing-in on small scales

Another examples: $\left\{\begin{array}{l}\text{singular behavior in} \\ \text{2D, closed-streamline flow}\end{array}\right.$

→ Homogenization Theory → $\left\{\begin{array}{l}\text{Prandtl, Batchelor} \\ \text{Weiss} \\ \text{Rhines, Young}\end{array}\right.$

recall $\underline{\omega}$ evolution for
$\underline{\nabla}\cdot\underline{v} = 0$

$$\frac{\partial}{\partial t}\underline{\omega} + \underline{v}\cdot\underline{\nabla}\omega = \underline{\omega}\cdot\nabla \underline{v} + \nu\nabla^2\omega$$

2D → $\underline{\omega}\cdot\nabla v = 0$ $\qquad \underline{\omega} = \omega(\hat{z})$

$v = \nabla\phi\times\hat{z}$

c.f. [P.A. Davidson 2004] – c.f.

then $\partial_t \omega + \nabla\phi\times\hat{z}\cdot\nabla\omega = \nu\nabla^2\omega$

more generally scalar ξ: $\left\{ \begin{array}{l} \text{active} \\ \text{or} \\ \text{passive} \end{array} \right.$

$$\partial_t \xi + \nabla\phi\times\hat{z}\cdot\nabla\xi = \nu\nabla^2\xi$$

Now: $t\to\infty$, $\partial_t\xi\to 0$

$$\nabla\phi\times\hat{z}\cdot\nabla\xi = \nu\nabla^2\xi$$

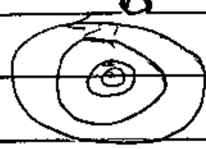
$\nu\to 0$ $\quad \nabla\phi\times\hat{z}\cdot\nabla\xi = 0$

i.e. $P_e \sim \frac{VL}{\nu}\to\infty$

$\sim Re$ $\quad \xi=\xi(\phi)$

i.e. bounded domain, closed streamline solution

ξ_0

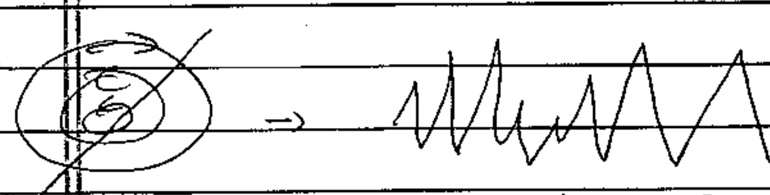


$\rightarrow \xi=\xi(\phi(r))$ is arbitrary solution

can develop arbitrarily fine scale $\xi(\phi)$ → closed streamlines ⇒ perfect memory

→ ∴ fine scale structure develops, no inter-streamline communication, & persists

i.e.



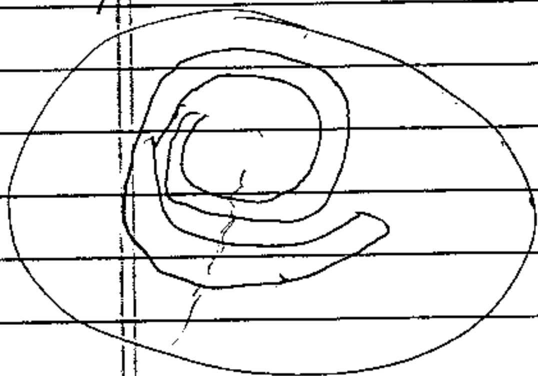
$\left\{ \begin{array}{l} \text{i.e. tag each} \\ \text{streamline,} \\ \text{arbitrarily} \end{array} \right.$

~ no smoothing of sharp gradients

"Not all solutions of the Navier-Stokes equations are realized in nature."
Landau & Lifshitz

→ i.e. of  → blob in concentric shear flow

blow-up



⇒ non-diffusive stretching produces arbitrarily fine scale structure!

now, point is that for $\nu \neq 0$
$Re, Pe \gg 1$

instead of arbitrarily fine scale structure

must have: $\omega(\psi) \to \text{const}$ as $t \to \infty$ — i.e. small ν ⇒ global behavior

⇒ i.e. finite ν at large Re ⇒ vorticity homogenization!, $\omega \to \text{const}$ within C_0

⇒ highly singular behavior!

$\nu = 0$ ⇒ Euler Eqn. (2D) → $\omega = \omega(\psi)$ solves

$\nu \neq 0$ ⇒ large Re 2D Navier-Stokes Eqn. → $\omega = \text{const}$ solves

note contrast!

Issues:

→ how long to homogenization? (what means asymptotic)

→ where is $\nabla\omega \neq 0$ ⇒ boundary layer thickness?

→ analogy in MHD? — Flux Expulsion

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J} \qquad \begin{aligned} \underline{v} &= \nabla\phi \times \hat{z} \\ \underline{B} &= \nabla A \times \hat{z} \end{aligned}$$

$$-\frac{1}{c}\partial_t \underline{A} - \nabla\phi + \frac{(\nabla\phi \times \hat{z})}{c} \times (\nabla A \times \hat{z}) = \eta \underline{J}$$

$\hat{z} \cdot (\quad)$

$$\Rightarrow -\frac{1}{c}\partial_t A_z - \cancel{\nabla\phi \cdot \hat{z}} + \hat{z} \cdot \left[\cancel{(\nabla\phi \times \hat{z}) \cdot \hat{z}}\right] \nabla A$$

$$- \frac{(\nabla\phi \times \hat{z}) \cdot \nabla A}{c} = \eta \underline{J}$$

$$\therefore \quad \partial_t \underline{A} + \nabla\phi \times \hat{z} \cdot \nabla A = \eta \nabla^2 A$$

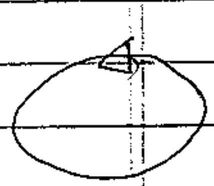
$$\Rightarrow \text{2 D convection} \begin{cases} \nabla \cdot \underline{v} = 0 \\ \eta \neq 0 \end{cases}$$

⇒ expect $\nabla A = 0$, except boundaries
$t \to \infty$

→ analogy is "flux expulsion"

E. Prandtl-Batchelor Theorem

* G. Batchelor, JFM 1 177 (1956) (posted)
P.B. Rhines and W.R. Young, JFM 122, 347 '82 (posted)
JFM 133, 133 '83
J. Pedlosky, "Ocean Circulation Theory"
see also Springer 1996, esp. 3.8.



Prandtl-Batchelor Theorem

Thm: Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by closed streamline C_0. Then, if diffusive dissipation,

i.e. $\partial_t \omega + \nabla\phi\times\hat{z}\cdot\nabla\omega = \nabla\cdot(\nu\nabla\omega)$

then vorticity → uniform (homogenization), as $t\to\infty$, within C_0.

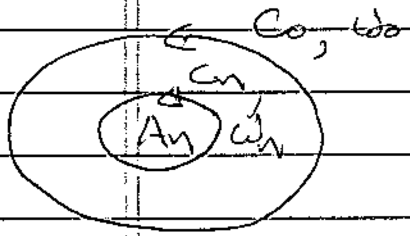
N.B.: ① finite $\nu \Rightarrow$ radically different final state

② no comment on "how long"!

$$-\nabla\phi\times\hat{z}\cdot\nabla\omega = \nabla\cdot\nu\nabla\omega$$

for stationarity [note $t\to\infty$ before $\nu\to 0$]

- choose arbitrary closed C_n within C_0.
  Here C_n a streamline



n.b. - assume simply connected region, i.e. no holes
- stationarity $\Rightarrow$ ω constant along streamlines.

$\therefore$ $\omega \to \omega_0$ on C_0 (ultimately C_0 sets b.c.)
$\omega \to \omega_n$ on C_n

if A_n is area enclosed by C_n;

$$\int_{A_n} d^2x\ \underline{v}\cdot\underline{\nabla}\,\omega = \int_{A_n} d^2x\ \underline{\nabla}\cdot\left(\nu\,\underline{\nabla}\omega\right)$$

but

$$\int_{A_n} d^2x\ \underline{v}\cdot\underline{\nabla}\,\omega = \int d^2x\ \underline{\nabla}\cdot[\underline{v}\,\omega]$$

$$= \oint_{C_n} dl\ \hat{n}_{C_n}\cdot(\underline{v}\,\omega)$$

$\vec{V} \to$ streamline, C_n

now

$\vec{\hat{n}}_{C_n} \to$ ~~normal to streamline C_n~~

so $\displaystyle\int_{C_n} d\ell \, (\hat{n}_{C_n} \cdot \underline{V}) \, \omega = 0$

as $\underline{V}$ is along streamline

$$0 = \int_{A_n} d^2x \; \underline{\nabla} \cdot (\nu \, \underline{\nabla} \omega)$$

$$= \nu \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla \omega$$

now in stationary state must have
$\omega \to$ const along streamline

$\therefore \; \omega = \omega(\phi)$

so $\omega_{C_n} = \omega(\phi_n)$

$$0 = \nu \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla \phi_n \, \frac{\delta \omega}{\delta \phi_n}$$

$$= \nu \frac{\delta \omega}{\delta \phi_n} \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla \phi_n$$

but $\Gamma = \int d\underline{\ell} \cdot \underline{v}$

$= \int d\underline{\ell} \cdot (\underline{\nabla}\phi \times \hat{z})$

$= \int (\hat{z} \times \vec{n}) \cdot (\underline{\nabla}\phi \times \hat{z})$

$= -\int d\ell \, (\underline{\nabla}\phi \cdot \vec{n}) = -\int d\ell \, (\nabla\phi \cdot \vec{n})$

$0 = \nu \dfrac{\partial \omega}{\partial \phi_n} \Gamma_n$

$\partial \omega / \partial \phi_n = 0$

but ϕ_n arbitrary $\Rightarrow$ $\partial \omega / \partial \phi = 0$, all ϕ

arbitrary $\Rightarrow$ no variation from line to line

$\Rightarrow$ $\boxed{\omega \text{ homogenized}}$!

so, expect $\nabla \omega$ larger at bounding contour C_0.

$\nabla \omega \to 0$, within $\Rightarrow$ $\nabla \omega$ held at boundary

Some Comments:

⇒ Homogenization theory looks 'magical' → caveat emptor!

i.e.

1.) note assumptions of

$t \to \infty$ ⇒ time asymptopia

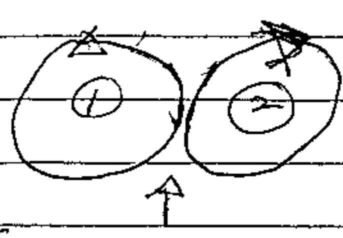
$q = q(\phi)$ ⇒ concentric streamlines

how long to achieve configuration?

2.) simply connected domain → annulus?

3.) single structure → expulsion from neighbors and possible interaction not addressed

i.e. what happens if? → 'interference of boundary layers'?



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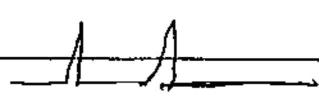


⇒ straining interaction

⇒ ② 'strips' ①, etc.

4.) Key Assumptions:

$\Rightarrow$ { closed, bounding streamline
{ viscous dissipation

i.e. can envision:

→ exact streamline, molecular viscosity

or

→ coarse-grained streamline, eddy viscosity

$\Rightarrow$ correspond to homogenization of

→ ~~total vorticity~~

→ mean/coarse-grained vorticity

⇔ time scales different

$\rightarrow \dfrac{\tau_{circulation}}{\tau_{diffusion}} \ll 1 \Rightarrow Re \gg 1$ (← L →)

– so establish concentric circulation lines

then

– diffuse across to homogenize → but slow ??

$$\frac{\tau_c}{\tau_d} = \frac{1}{(V/L)} \frac{D}{L^2} \ll 1 \Rightarrow \frac{D}{VL} \ll 1$$

i.e. $Re \gg 1$

or equivalently $\frac{VL}{D} >> 1$

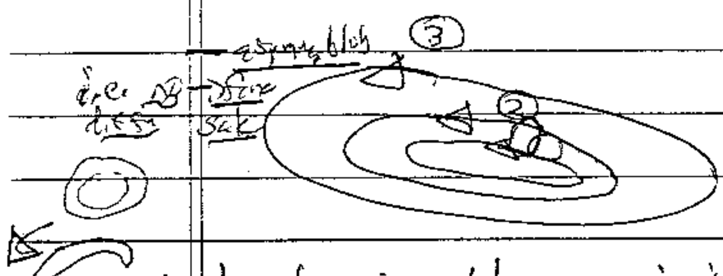
i.e. $(Re)_{eff,\,cell} >> 1$ ⟺ $Pe >> 1$

related: – essential idea is that $\bar{q}$ constant along streamlines established on fast ($\sim\tau_c$) scale
few

– dissipation homogenizes on slower ($\sim\tau_D$) time scale
(but this is slow ...)

→ What are the time scales? – $\left\{\begin{array}{l}\text{how resolve} \\ \text{slow time} \\ \text{scale problem}\end{array}\right.$

– useful to consider differentially rotating sheared flow within closed pattern



$V_1 \neq V_2 \neq V_3$

need blob with finite l_y

What is the mixing time scale?

1 finer, finer scale ...

2 Key: synergism between $\left\{\begin{array}{l}\text{shear} \\ \text{diffusion}\end{array}\right.$!

c.f. $\left\{\begin{array}{l}\text{H. Biglari, P.H. Diamond, P.W. Terry} \\ \text{Phys Fluids (B2), 1, 1990}\end{array}\right.$
(first noted by G.I. Taylor)

mixing Shear Dispersion

i.e. compare  radial diffusion

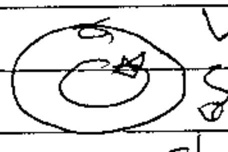
time $\Delta \; L \Rightarrow$

(a) pure diffusion

$$1/\tau \sim D/L^2 \qquad D_r \sim \langle \tilde{V}_r^2 \rangle \tilde{\tau}_c$$

$$\boxed{\langle \delta r^2 \rangle \sim D_r t}$$ → usual diffusion   any radial scattering process (unspecified)

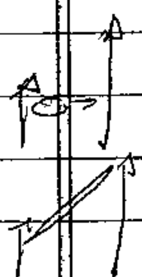
with

(b) diffusion + shear hybrid 

$$V' \sim V/L$$

shear

now $\frac{dr}{dt} = \tilde{V}_r$ → random walk



$$r\theta = y$$

$$\frac{dy}{dt} = V_{0}(r) \quad \rightarrow \quad \text{streaming}$$

shear

$$\therefore \frac{d}{dt}\delta y = \left(\frac{\partial V_{0y}}{\partial r}\right)\delta r$$

$$\delta y = \int \left(\frac{\partial V_{0y}}{\partial r}\right)\delta r \, dt$$

$$\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 \langle \delta r^2 \rangle t^2$$

$$\langle \delta r^2 \rangle \sim D_r t$$

shear dispersion

$$\Rightarrow \boxed{\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 D_r t^3}$$ → hybrid decorrelation $\langle \delta y^2 \rangle \sim t^3$

scale of comparison

↓

$\langle \delta y^2 \rangle \sim L_y^2 \Rightarrow$ arbitrary

$$1/\tilde{\tau}_{mix} = \left(\left(\frac{\partial V_y}{\partial x}\right)^2 \frac{D}{L_y^2} \right)^{1/3}$$

$$\sim \left(\left(\frac{V_0}{L_0}\right)^2 \frac{D}{L_y^2} \right)^{1/3}$$

$$\sim \frac{V_0}{L_y} \left(\frac{D}{V L_0} \right)^{1/3}$$

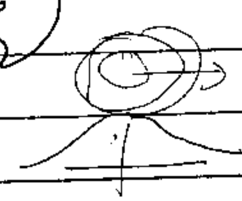
$\therefore \quad 1/\tilde{\tau}_{mix} \sim \frac{1}{\tau_0} (Re)^{-1/3}$

Re >> 1 by construction → consistent ✓

so have:

→ mixing / homogenization on hybrid time scale → time to come to ⓞ symmetric state

$$1/\tilde{\tau}_{mix} = 1/\tau_0 \left(\tau_0 / \tau_D \right)^{1/3}$$



only radial diffn possible

$\Rightarrow \frac{1}{\tau_0} > \frac{1}{\tilde{\tau}_{mix}} > \frac{1}{\tau_D}$ time to uniformize vs τ_D

⇒ PV homogenization most relevant to closed eddys with sheared rotation

Some Points

i.) Time scales

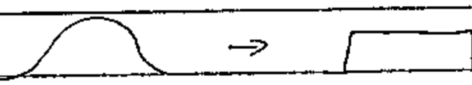
have Re, $Pe >> 1 \Rightarrow \frac{\tau_D}{\tau_c} >> 1$

but

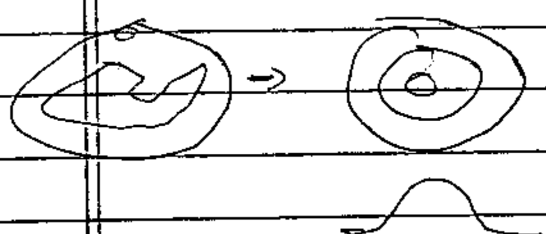
$\tau_{mix} \sim Re^{1/3} \tau_c \sim \frac{\tau_D}{Re^{2/3}}$

$\tau_{mix} < \tau_D$ → time to homogenize

time to establish $\sim$ azimuthally symmetric state



i.e.



but radially profiled

ii.) Point of theorem is global impact of small dissipation.

iii.) Interesting to note that P-B theorem applies to both active, passive scalar.

ii) Observe: all that is really required for applicability of theory is:

- → incompressible advection – 2D: $\nabla\phi\times\hat{z}\cdot\nabla$
- → closed streamline → { fine, coarse }
- → diffusive dissipation → { molecular, eddy }

i.e. can apply to magnetic potential, as noted previously, i.e.

2D MHD $\quad \underline{U} = \nabla\phi\times\hat{z}$, $\underline{B} = \nabla A\times\hat{z}$

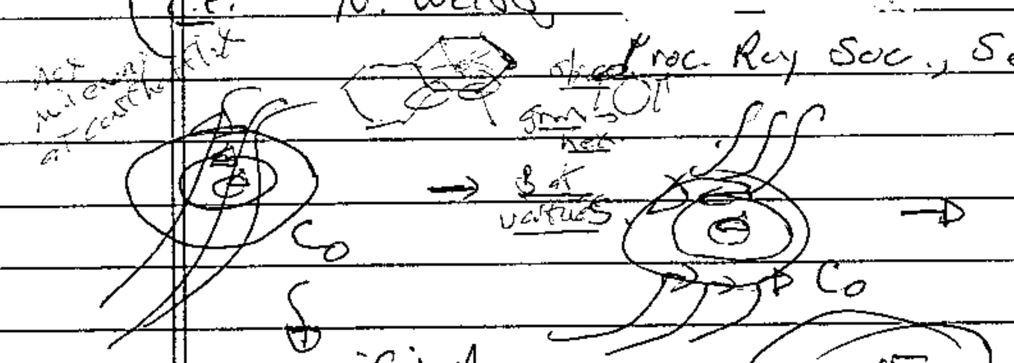
$$\frac{\partial A}{\partial t} + \underline{V}\cdot\nabla A = \eta\nabla^2 A$$

$E + \underline{v}\times B = \eta J$

$-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\phi + (\nabla\phi\times\hat{z})\times(\nabla A\times\hat{z}) = \eta J$

$\hat{z}\cdot\left[\left(-\frac{1}{c}\frac{\partial A}{\partial t}\right) + (\nabla\phi\times\hat{z})\cdot\hat{z} - (\nabla\phi\times\hat{z}\cdot\nabla A)\hat{z}\right] = \eta\nabla$

→ Famous problem of Flux Expulsion (i.e. N. Weiss, Proc. Roy Soc., Series A 293, 310, 1966)



→ $\nabla A \to 0$ within

→ B expelled to boundary (i.e. $B = \nabla A\times\hat{z}$)

⇒ $B \sim 0$ in cell.

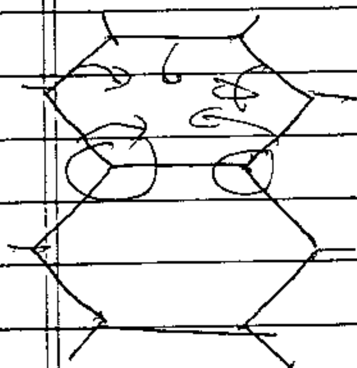
specified convective cell (has magnetoconvection as aim)

see 15a

→ obvious that above argument can be recycled, so

$\delta A_n / \delta\phi_n = 0$, for all C_n in C_0.

i.e. top view → solar granulation

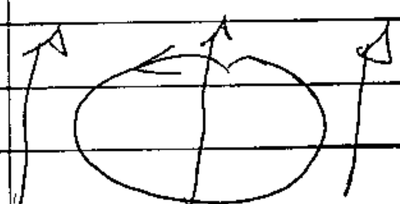


~ hexagonal pattern

~ field strongest at vertices

~ expulsion

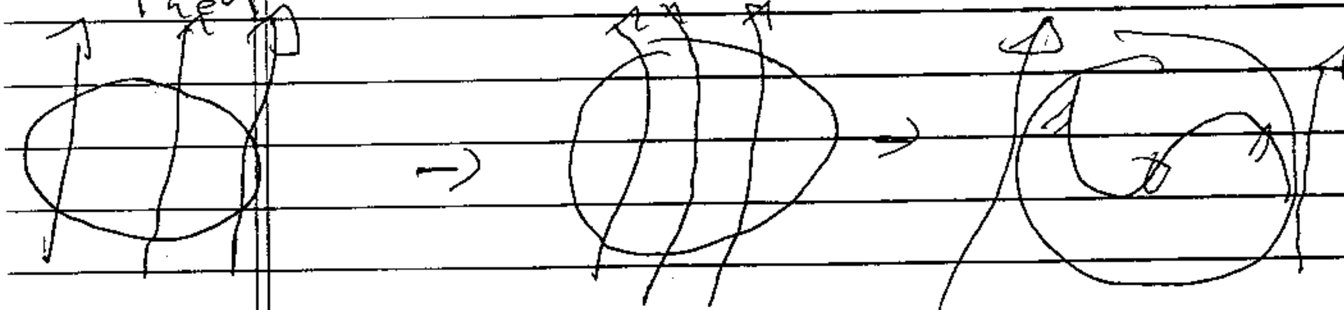
suggests → side view → toy problem of flux expulsion



→ i.e.

cell in uniform field

then



see Weiss, Proc. Roy Soc A 293, 310-1966

also Moffatt, 3.7 – 3.10

→ here requirement is $R_m = \frac{LV}{\eta} \gg 1$

and

$$1/\tau_{hang} \sim 1/\tau_{mix} \sim \frac{V_0}{L_0}(R_m)^{-1/3}$$

↓

time scale for flux expulsion

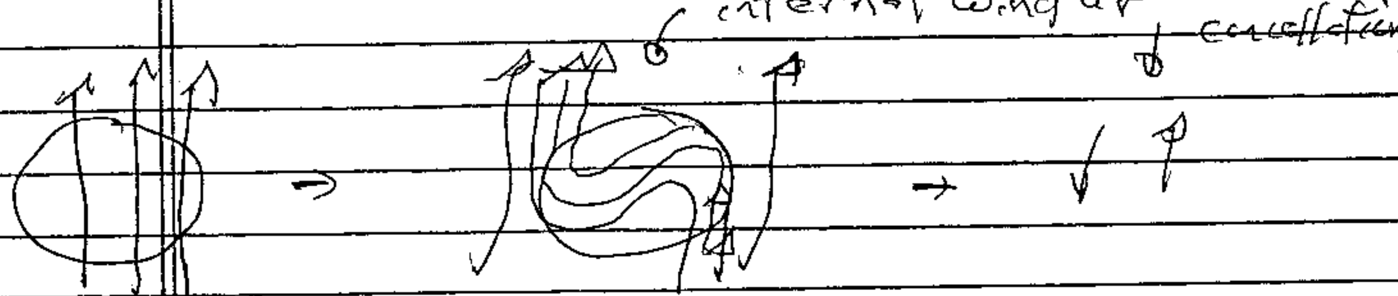
→ physically, can see relevant time scale by noting:

→ wind up must conserve volume/mass

→ wind up must conserve flux

→ irreversibility sets in when $R_m|_{local} \sim 1$

⇒ dissipation of field matches drive by wind-up



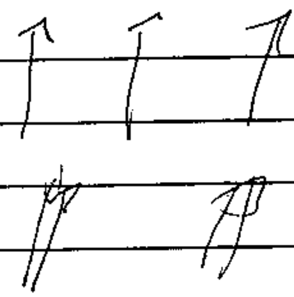
internal cancellation → { field expulsion
flux homogenization

80

- Field stretched / compressed in s-wards

$n\ell = L_0$

$\ell = L_0/n$



- Flux (vertical) conserved, so

so $B_0 L \sim l B$ ($B \uparrow$ upon flux conservation/winding)

$\Rightarrow$ $B \sim n B_0$

$\Rightarrow$ now expect freezing-in lost when compressed field

$$(R_m)_{eff} \sim 1 \Rightarrow \frac{V B_0}{L_0} \sim \eta \frac{B}{l^2} \sim \eta \frac{n B_0}{L_0^2/n^2}$$

$\Rightarrow$

$$n^3 \sim \left(\frac{V_0 L_0}{\eta}\right) \Rightarrow \boxed{n \sim R_m^{1/3}}$$

# of turns to render boundary diffusive

~~thickness of B layer~~

$$\frac{V B}{L} \sim \frac{\eta B}{\delta^2} \qquad \frac{\delta}{L} \sim 1/\sqrt{R_m}$$

but: - diffusion in boundary layer $\Leftrightarrow$ homogenization within

so

- $n \sim R_m^{1/3}$ $\Rightarrow$ # turns for homogenization

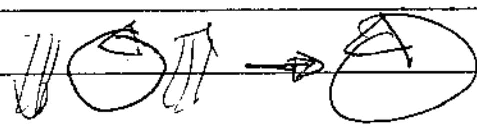
- $\tilde{\tau}_{hom} \sim \tilde{\tau}_0 R_m^{1/3}$ $\rightarrow$ time ✓

agree above

⇒ expect flux expelled from closed cell
→ field strongest at boundary

→ possibly explain why strongest cells in 2D

magneto convection often
reach a state independent of B
while neighbors quenched:



Now: ⇒ why care about this?
Why homogenization important
→ similar phenomena

① identifies a trend; i.e. in spirit of
Taylor Theory (E_{mag} minimized s/t
$\int \underline{A} \cdot \underline{B} \, d^3x$ conserved), homogenization
theory identifies a trend, i.e.

if f
- conserved locally by 2D flow, $\underline{\nabla} \cdot \underline{v} = 0$
- diffused
- enclosed

⇒ f homogenized

~~→ non-passive → vorticity~~
② trend applies to ~~→ passive~~ → A, C

In particular,

③ trend severely constrains form of vorticity flux, flow evolution

i.e. zonal flows ⇒ 2D closed streamline flows ....

→ Do zonal flows tend homogenize PV?

→ if noise (i.e. emission), what scale selected?